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On the derivative of meromorphic functions with multiple zeros ☆

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Abstract

Let f be a transcendental meromorphic function and let R be a rational function, $R \not\equiv 0$. We show that if all zeros and poles of f are multiple, except possibly finitely many, then $f' - R$ has infinitely many zeros. If f has finite order and R is a polynomial, then the conclusion holds without the hypothesis that poles be multiple.

1. Introduction and results

Let F be a transcendental meromorphic function, $c \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$. (Here and in the following, unless stated otherwise, "meromorphic" always means "meromorphic in the complex plane $\mathbb{C}$.") Hayman [5, Corollary to Theorem 9] proved that if $n \geqslant 3$, then $F'F^n - c$ has infinitely many zeros. He conjectured that this also holds for $n = 1$ and $n = 2$. This conjecture was confirmed by Mues [12, Satz 3] for the case $n = 2$ and finally the case $n = 1$ was settled in [2,3,20]. Actually the method of [2,3,20] applies for all $n \in \mathbb{N}$.

The structure of the proof of Hayman's conjecture in [2,3,20] is as follows. First it was proved in [2] that the conjecture is true for functions of finite order. Then normal family

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arguments (cf. Lemma 2.1 below) were used to reduce the general case to the finite order case.

We note that if $n \in \mathbb{N}$ and $f := \frac{1}{n+1}F^{n+1}$, then $f' = F'F^n$, and f has only multiple zeros and poles. It turns out that some results concerning functions of the form $F'F^n$ hold more generally for derivatives of functions with multiple zeros.

In the case of finite order we have the following results.

Theorem A [2, Theorem 3]. *Let f be a meromorphic function of finite order and $c \in \mathbb{C}\backslash\{0\}$. If f has infinitely many multiple zeros, then $f' - c$ has infinitely many zeros.*

Theorem B [18, Lemma 6]. *Let f be a transcendental meromorphic function of finite order and $c \in \mathbb{C}\backslash\{0\}$. If f has only multiple zeros, then $f' - c$ has infinitely many zeros.*

While it was shown in [2, p. 370] that the hypothesis that f be of finite order cannot be omitted in Theorem A, we do not know whether it is necessary in Theorem B.

For functions of unrestricted growth, we have the following result.

Theorem C [18, Theorem 1]. *Let f be a transcendental meromorphic function and $c \in \mathbb{C}\backslash\{0\}$. If f has only multiple zeros and poles, then $f' - c$ has infinitely many zeros.*

A discussion of the case where f is rational leads to the following result.

Theorem D (cf. [18, Lemma 9]). *Let f be a meromorphic function and $c \in \mathbb{C}\backslash\{0\}$. If f has only multiple zeros and poles, and $f' - c$ has no zeros, then f is constant.*

While it is not known whether the hypothesis that poles be multiple is necessary in Theorem C, it cannot be omitted in Theorem D, as shown by the example $f(z) = c(z-1)^2/z$.

The question whether the constant c in the above results can be replaced by a rational function was addressed in [1]. It was shown in [1] that if F is a transcendental meromorphic function of finite order and P is a polynomial which does not vanish identically, then $F'F - P$ has infinitely many zeros. The method used also shows that $F'F^n - P$ has infinitely many zeros for every $n \in \mathbb{N}$. Here we remove the restriction on the order and also allow a rational function instead of a polynomial.

Theorem 1.1. *Let f be a transcendental meromorphic function and let R be a rational function, $R \not\equiv 0$. Suppose that all zeros and poles of f are multiple, except possibly finitely many. Then $f' - R$ has infinitely many zeros.*

It seems reasonable to conjecture that the conclusion of Theorem 1.1 holds without the hypothesis that the poles be multiple. In this direction, we have the following result.

Theorem 1.2. *Let f be a transcendental meromorphic function of finite order and let P be a polynomial, $P \not\equiv 0$. Suppose that all zeros of f are multiple, except possibly finitely many. Then $f' - P$ has infinitely many zeros.*

2. Proof of Theorem 1.1

The main tool in the proof of Theorem 1.1 is the following result.

Lemma 2.1. *Let $\mathcal{F}$ be a family of functions meromorphic in a domain $D \subset \mathbb{C}$ and let $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ with $-m < \alpha < 1$. Suppose that the zeros of the functions in $\mathcal{F}$ have multiplicity at least m and that $\mathcal{F}$ is not normal at $z_0 \in D$. Then there exist a sequence (f_k) in $\mathcal{F}$, a sequence (z_k) in D, a sequence (ρ_k) of positive real numbers and a non-constant function f which is meromorphic in $\mathbb{C}$ such that $z_k \to z_0$, $\rho_k \to 0$ and*

$$\rho_k^{\alpha} f_k(z_k + \rho_k z) \to f(z)$$

locally uniformly in $\mathbb{C}$.

Moreover, the spherical derivative $f^{\#} := |f'|/(1 + |f|^2)$ of f satisfies $f^{\#}(z) \leqslant f^{\#}(0) = 1$ for all $z \in \mathbb{C}$. In particular, f has finite order.

The case $\alpha = 0$ of this lemma is due to Zalcman [19]. Pang [14,15] proved that one can always take $-1 < \alpha < 1$. Pang and Xue [16] showed that $\alpha < 0$ is admissible if the functions in $\mathcal{F}$ have no zeros. The above version is due to Chen and Gu [4, Theorem 2]. For a survey of applications of this lemma we refer to [21].

Next we recall that a meromorphic function g is called a *Julia exceptional function* if $g^{\#}(z) = \mathrm{O}(1/|z|)$ as $|z| \to \infty$.

Lemma 2.2. *Let g be a meromorphic function which is not a Julia exceptional function. Then there exists a sequence (a_k) in $\mathbb{C}$ such that $a_k \to \infty$, $a_k g'(a_k) \to \infty$ and $g(a_k) \to 0$ as $k \to \infty$.*

Proof. Since g is not a Julia exceptional function, there exists a sequence (b_n) in $\mathbb{C}$ such that $b_n \to \infty$ and $b_n g^{\#}(b_n) \to \infty$. We define $g_n(z) := g(b_n z)$. Then $g_n^{\#}(1) = |b_n| g^{\#}(b_n) \to \infty$ so that (g_n) is not normal at 1. Using Lemma 2.1 for $\alpha = 0$ we obtain sequences (n_k), (z_k) and (ρ_k) satisfying $n_k \in \mathbb{N}$, $n_k \to \infty$, $z_k \in \mathbb{C}$, $z_k \to 1$, $\rho_k > 0$ and $\rho_k \to 0$ such that

$$g_{n_k}(z_k + \rho_k z) \to h(z)$$

for some non-constant function h meromorphic in $\mathbb{C}$. Given $\varepsilon > 0$, there exists $\xi \in \mathbb{C}$ with $|h(\xi)| < \varepsilon$ and $h'(\xi) \neq 0$. With $c_k := (z_k + \rho_k \xi) b_{n_k}$ we have

$$g(c_k) = g_{n_k}(z_k + \rho_k \xi) \to h(\xi)$$

as $k \to \infty$. Moreover,

$$\rho_k b_{n_k} g'(c_k) = \rho_k g_{n_k}'(z_k + \rho_k \xi) \to h'(\xi)$$

as $k \to \infty$. Since $\rho_k \to 0$ and $z_k \to 1$ we have $c_k \sim b_{n_k}$ as $k \to \infty$. This yields

$$c_k g'(c_k) = \bigl(1 + \mathrm{o}(1)\bigr) \frac{h'(\xi)}{\rho_k} \to \infty$$

as $k \to \infty$. Altogether we have thus found a sequence (c_k) in $\mathbb{C}$ such that $c_k \to \infty$, $c_k g'(c_k) \to \infty$ and $\limsup_{k\to\infty} |g(c_k)| \leqslant \varepsilon$. Since ε can be chosen arbitrarily small, we deduce that a sequence (a_k) with the properties stated exists. □

Next we need the following result of Lehto and Virtanen [11, p. 7].

Lemma 2.3. *A transcendental Julia exceptional function does not have an asymptotic value.*

We shall use some standard terminology and results from Nevanlinna theory; see [7,10, 13]. It follows easily from the Ahlfors–Shimizu form of the Nevanlinna characteristic that if f is a Julia exceptional function, then $T(r, f) = \mathrm{O}((\log r)^2)$ as $r \to \infty$.

We shall need the following two results concerning functions satisfying this growth condition. The first one is due to Hayman [6, Corollary to Theorem 1]; see also his book [8, p. 442, Corollary 4].

Lemma 2.4. *Let h be an entire function satisfying*

$$\log M(r, h) = \mathrm{O}\big((\log r)^2\big) \tag{1}$$

as $r \to \infty$. Then $\log|h(re^{i\theta})| \sim \log M(r, h)$ as $r \to \infty$ for almost every $\theta \in [0, 2\pi]$.

We shall use Lemma 2.4 to prove the following result.

Lemma 2.5. *Let f be a transcendental meromorphic function and let R be a rational function satisfying $R(z) \sim cz^d$ as $z \to \infty$, with $c \in \mathbb{C}\backslash\{0\}$ and $d \in \mathbb{Z}$. Suppose that $f' - R$ has only finitely many zeros and that $T(r, f) = \mathrm{O}((\log r)^2)$ as $r \to \infty$. Define $g(z) := f(z)/z^{d+1}$, with $g := f$ if $d = -1$. Then g has an asymptotic value.*

Proof. Since $f' - R$ has only finitely many zeros, there exists a polynomial $P \not\equiv 0$ such that $h := P/(f' - R)$ is entire. By standard results in Nevanlinna theory, we have

$$\log M(r, h) \leqslant 3T(2r, h) \leqslant 3T(2r, f') + \mathrm{O}(\log r)$$

and

$$T(r, f') \leqslant 2T(r, f) + m\left(r, \frac{f'}{f}\right) = 2T(r, f) + \mathrm{O}(\log r)$$

as $r \to \infty$. Thus h satisfies (1). Let $m := \deg(P) + 2 + |d|$. By Lemma 2.4 there exists $\theta \in [0, 2\pi]$ such that $|h(re^{i\theta})|r^{-m} \to \infty$ as $r \to \infty$. Hence

$$\big|f'(re^{i\theta}) - R(re^{i\theta})\big| = \left|\frac{P(re^{i\theta})}{h(re^{i\theta})}\right| \leqslant \frac{1}{r^{2+|d|}}$$

for sufficiently large r, say $r \geqslant r_0$. It follows that

$$\int_{r_0}^{r} \big(f'(te^{i\theta}) - R(te^{i\theta})\big)\, dt$$

tends to a finite limit as $r \to \infty$. If $d \geqslant 0$ we obtain

$$f\left(re^{i\theta}\right) \sim \frac{c}{d+1}\left(re^{i\theta}\right)^{d+1}$$

as $r \to \infty$. Thus $g(re^{i\theta}) \to c/(d+1)$ as $r \to \infty$. If $d \leqslant -2$ we obtain

$$f\left(re^{i\theta}\right) = a + \frac{c}{d+1}\left(re^{i\theta}\right)^{d+1} + \mathrm{O}\left(r^d\right)$$

for some $a \in \mathbb{C}$ as $r \to \infty$. If $a = 0$ we find again that $g(re^{i\theta}) \to c/(d+1)$ as $r \to \infty$, while $g(re^{i\theta}) \to \infty$ if $a \neq 0$. Finally, if $d = -1$, then $f(re^{i\theta}) \sim c \log r$ so that $g(re^{i\theta}) = f(re^{i\theta}) \to \infty$ as $r \to \infty$. $\square$

Proof of Theorem 1.1. We assume that $f' - R$ has only finitely many zeros. We choose c, d and g as in Lemma 2.5; that is, $R(z) \sim cz^d$ as $z \to \infty$ and $g(z) = f(z)/z^{d+1}$. First we assume that g is a Julia exceptional function. Then $T(r, g) = \mathrm{O}((\log r)^2)$ and hence $T(r, f) = \mathrm{O}((\log r)^2)$ as $r \to \infty$, and thus g has an asymptotic value by Lemma 2.5. This is a contradiction to Lemma 2.3.

Thus g is not a Julia exceptional function and hence there exists a sequence (a_k) as in Lemma 2.2. We then have

$$\frac{f(a_k)}{a_k^{d+1}} = g(a_k) \to 0 \tag{2}$$

and

$$\frac{f'(a_k)}{a_k^d} = a_k g'(a_k) + (d+1)g(a_k) \to \infty \tag{3}$$

as $k \to \infty$.

First we consider the case that $d \neq -1$. For $D := \{z \in \mathbb{C}\colon |z-1| < \frac{1}{2}\}$ and $\mu := \frac{1}{d+1}$ we consider the function $h_k : D \to \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ defined by

$$h_k(z) = \frac{f(a_k z^\mu)}{c\mu a_k^{d+1}}.$$

Here z^μ denotes the branch of the root that fixes 1. We have

$$h_k'(z) = \frac{f'(a_k z^\mu) z^\mu}{c a_k^d z} \neq \frac{R(a_k z^\mu) z^\mu}{c a_k^d z}$$

if $z \in D$ and k is sufficiently large. With

$$s_k(z) := \frac{R(a_k z^\mu) z^\mu}{c a_k^d z}$$

we thus have

$$h_k'(z) \neq s_k(z) \tag{4}$$

if $z \in D$ and k is sufficiently large. By the definition of c, d and μ we have

$$s_k(z) \to 1 \tag{5}$$

as $k \to \infty$, uniformly for $z \in D$. By (2) and (3) we have

$$h_k(1) = \frac{f(a_k)}{c\mu a_k^{d+1}} \to 0$$

and

$$h_k'(1) = \frac{f'(a_k)}{c a_k^d} \to \infty$$

as $k \to \infty$. Thus $h_k^{\#}(1) \to \infty$ as $k \to \infty$. This implies that (h_k) is not normal at 1. For sufficiently large k all zeros and poles of h_k in D are multiple. Thus we can apply Lemma 2.1 with $\alpha = -1$ and obtain sequences $(k_j), (z_j)$ and (ρ_j) satisfying $k_j \in \mathbb{N}$, $k_j \to \infty$, $z_j \in D$, $z_j \to 1$, $\rho_j > 0$ and $\rho_j \to 0$ such that

$$\frac{h_{k_j}(z_j + \rho_j z)}{\rho_j} \to h(z)$$

for some non-constant function h meromorphic in $\mathbb{C}$. By Hurwitz's theorem, h has only multiple zeros and poles. Since $h_{k_j}'(z_j + \rho_j z) \to h'(z)$ we deduce from (4), (5) and Hurwitz's theorem that $h' - 1$ has no zeros. This contradicts Theorem D.

We now consider the case that $d = -1$. Here we define $D := \{z \in \mathbb{C}\colon |z| < 1\}$ and $h_k : D \to \widehat{\mathbb{C}}$ by

$$h_k(z) = \frac{f(a_k e^z)}{c}.$$

Then

$$h_k'(z) = \frac{f'(a_k e^z) a_k e^z}{c}$$

and with

$$s_k(z) = \frac{R(a_k e^z) a_k e^z}{c}$$

we find again that (4) and (5) hold. Similarly as before we have $h_k^{\#}(0) \to \infty$, and an application of Lemma 2.1 leads again to a contradiction. □

3. Proof of Theorem 1.2

We shall use arguments similar to those used in [1]. As in [1] we need the following result proved in [2, Corollary 3].

Lemma 3.1. *Let g be a meromorphic function of finite order. If g has only finitely many critical values, then g has only finitely many asymptotic values.*

The next result is due to Rippon and Stallard [17, Lemma 2.2].

Lemma 3.2. *Let g be a transcendental meromorphic function and suppose that the set of all finite critical and asymptotic values of g is bounded. Then there exists $R > 0$ such that if $|z| > R$ and $|g(z)| > R$, then*

$$\left|g'(z)\right| \geqslant \frac{|g(z)| \log |g(z)|}{16\pi |z|}.$$

Finally we need the following lemma which follows from a result of Hua [9].

Lemma 3.3. *Let f be a transcendental meromorphic function and let P be a polynomial, $P \not\equiv 0$. Then at least one of the function f and $f' - P$ has infinitely many zeros.*

This extends a classical result of Hayman (see [5, Theorem 3] or [7, Corollary to Theorem 3.5]) dealing with the case that P is constant.

Proof of Theorem 1.2. We choose a polynomial Q such that $Q' = P$ and define $g := f - Q$. We assume that $g' = f' - P$ has only finitely many zeros. Then g has only finitely many asymptotic values by Lemma 3.1, and thus g satisfies the hypotheses of Lemma 3.2. It follows from Lemma 3.3 that f has infinitely many zeros, say $f(z_k) = 0$, with $z_k \to \infty$ as $k \to \infty$. We clearly have $g(z_k) = -Q(z_k)$. Since f has only finitely many simple zeros, z_k is a multiple zero of f and hence $g'(z_k) = -Q'(z_k)$ for large k. Lemma 3.2 yields

$$\left|Q'(z_k)\right| = \left|g'(z_k)\right| \geqslant \frac{|g(z_k)| \log |g(z_k)|}{16\pi |z_k|} = \frac{|Q(z_k)| \log |Q(z_k)|}{16\pi |z_k|}$$

and thus

$$\frac{|z_k Q'(z_k)|}{|Q(z_k)|} \geqslant \frac{\log |Q(z_k)|}{16\pi}$$

for large k. This is a contradiction, since the left side of the last inequality tends to $\deg(Q)$ as $k \to \infty$, while the right side tends to ∞. □

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